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# A generating function for Laguerre–Sobolev orthogonal polynomials

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**Abstract**

Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to the Sobolev inner product

$$(f,g)_S = \int_0^{+\infty} f(x)g(x)x^\alpha e^{-x}\,dx + \lambda \int_0^{+\infty} f'(x)g'(x)x^\alpha e^{-x}\,dx,$$

where $\alpha > -1$, $\lambda > 0$ and the leading coefficient of the $S_n$ is equal to the leading coefficient of the Laguerre polynomial $L_n^{(\alpha)}$. In this work, a generating function for the Sobolev–Laguerre polynomials is obtained.

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## 1. Introduction

Consider the Sobolev inner product

$$(f,g)_S = \int_0^{+\infty} f(x)g(x)x^\alpha e^{-x}\,dx + \lambda \int_0^{+\infty} f'(x)g'(x)x^\alpha e^{-x}\,dx, \tag{1}$$

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with $\alpha > -1$ and $\lambda > 0$. Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to (1), normalized by the condition that $S_n$ and the Laguerre polynomial $L_n^{(\alpha)}$ have the same leading coefficient ($n = 0, 1, 2, \ldots$).

The special case $\alpha = 0$ has already been studied by Brenner [1]. In [11], Schäfke and Wolf introduced *einfache verallgemeinerte klassische Orthogonalpolynome* and the above defined sequence $\{S_n\}$ is a special case of them. The inner product (1) can also be studied as a special case of inner products defined by a *coherent pair of measures* as introduced by Iserles et al. [4]. For a survey of possible applications and results on Sobolev orthogonal polynomials, see [5,9].

The most complete treatment of the sequence $\{S_n\}$ orthogonal with respect to (1) appears in a paper of Marcellán et al. [7]. The paper gives among others several algebraic and differential relations with $\{L_n^{(\alpha)}\}$, a four-term recurrence relation, a Rodrigues-type formula and some properties concerning the zeros. An asymptotic result for $S_n(x)$ with $x \in \mathscr{C} \setminus [0, +\infty)$ and $n \to \infty$, has been obtained by Marcellán et al. [6] in a recent paper.

Finally, we remark that asymptotic results for polynomials orthogonal with respect to a Sobolev inner product defined by a coherent pair of measures has been derived by Martínez-Finkelshtein et al. [8] in the Jacobi case and by Meijer et al. [10] in the Laguerre case.

The aim of the present paper is to derive a generating function for the Laguerre–Sobolev polynomials. Our result is a generalization of the generating function for the Laguerre polynomials $L_n^{(\alpha)}$

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\omega^n = (1-\omega)^{-\alpha-1} \exp\left(-\frac{x\omega}{1-\omega}\right) \tag{2}$$

(see Szegő [12, p. 101, (5.1.9)]). The particular case $\alpha = 0$ has been studied by Wimp and Kiesel [13] with a different technique.

Section 2, gives the basic relations on Laguerre–Sobolev polynomials. In particular, it is shown that a generating function for the Laguerre–Sobolev polynomials can be found from a generating function for the classical Laguerre polynomials (Lemma 2.5). As a consequence, we refind the result of Wimp and Kiesel (Theorem 2.1). In Section 3, a generating function for Laguerre–Sobolev polynomials if $\alpha \neq 0$ is derived. The main result is stated in Theorem 3.1. Finally, in Section 4, some generalizations are discussed.

## 2. Laguerre–Sobolev orthogonal polynomials

Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to the Sobolev inner product

$$(f, g)_S = \int_0^{+\infty} f(x)g(x)x^{\alpha}e^{-x}\,dx + \lambda \int_0^{+\infty} f'(x)g'(x)x^{\alpha}e^{-x}\,dx, \tag{3}$$

with $\alpha > -1$ and $\lambda > 0$. The $S_n$ are normalized by the condition that the leading coefficient of $S_n$ equals the leading coefficient of $L_n^{(\alpha)}$.

Observe that $S_0 = L_0^{(\alpha)}$ and $S_1 = L_1^{(\alpha)}$.

Several authors obtained the following result, see e.g. [7].

**Lemma 2.1.** *There exist positive constants $a_n$ depending on $\alpha$ and $\lambda$, such that*

$$L_n^{(\alpha-1)}(x) = L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) = S_n(x) - a_{n-1}S_{n-1}(x), \quad n \geqslant 1. \tag{4}$$

Marcellán et al. [7] found the following recurrence relation.

**Lemma 2.2.** *The sequence $\{a_n\}$ in* (4) *satisfies*

$$a_n = \frac{n+\alpha}{n(2+\lambda)+\alpha-na_{n-1}}, \quad n \geqslant 1 \tag{5}$$

*with*

$$a_0 = 1.$$

In order to derive a generating function for $S_n$ we need more information on the sequence $\{a_n\}$.

**Lemma 2.3.** *The sequence $\{a_n\}$ is convergent, and*

$$a = \lim_{n\to\infty} a_n = \frac{\lambda+2-\sqrt{\lambda^2+4\lambda}}{2} < 1. \tag{6}$$

**Proof.** First, we observe that a simple induction argument applied on Lemma 2.2 gives $a_n \leqslant 1$ for all $n \geqslant 0$.

Suppose that $a = \lim_{n\to\infty} a_n$ exists, then (5) implies

$$a^2 - a(2+\lambda) + 1 = 0.$$

Since $a_n \leqslant 1$ for all $n \geqslant 0$, we have $a \leqslant 1$. Hence

$$a = \frac{\lambda+2-\sqrt{\lambda^2+4\lambda}}{2} < 1.$$

Now, we prove that $\{a_n\}$ is indeed convergent to $a$.

With (5) and $a(2+\lambda) = a^2+1$ we have

$$a_n - a = \frac{\alpha - \alpha a + na(a_{n-1}-a)}{n(2+\lambda)+\alpha-na_{n-1}}.$$

Then, using $a_{n-1} \leqslant 1$,

$$|a_n - a| \leqslant \frac{|\alpha-\alpha a|}{n(1+\lambda)+\alpha} + \frac{na|a_{n-1}-a|}{n(1+\lambda)+\alpha}.$$

Hence

$$\limsup |a_n - a| \leqslant \frac{a}{1+\lambda} \limsup |a_n - a|.$$

Since $\frac{a}{1+\lambda} < 1$, the lemma follows. $\square$

From the sequence $\{a_n\}$ we construct a sequence $\{q_n(\lambda)\}$ of polynomials in $\lambda$.

**Lemma 2.4.** *Define the sequence* $\{q_n(\lambda)\}$ *by*

$$q_0(\lambda) = 1, \quad q_{n+1}(\lambda) = \frac{q_n(\lambda)}{a_n}, \quad n \geqslant 0.$$

*Then* $q_n(\lambda)$ *is a polynomial in* $\lambda$, $\deg q_n = n-1$ *if* $n \geqslant 1$, *satisfying the three-term recurrence relation*

$$(n+\alpha)q_{n+1}(\lambda) = (n(\lambda+2)+\alpha)q_n(\lambda) - nq_{n-1}(\lambda), \quad n \geqslant 1 \tag{7}$$

*with initial conditions* $q_0(\lambda) = q_1(\lambda) = 1$.

**Proof.** The recurrence relation (7) is just relation (5) rewritten in terms of $q_n$. Since $a_0 = 1$, $q_1 = 1$ and then (7) implies that, for $n \geqslant 1$, $q_n$ is a polynomial in $\lambda$ of degree $n-1$. $\square$

The convergence of a series involving the Laguerre–Sobolev orthogonal polynomials can be reduced to the convergence of a series involving Laguerre polynomials.

**Lemma 2.5.** *For* $|\omega| < a < 1$ *we have*

$$\sum_{n=0}^{\infty} q_n(\lambda) S_n(x) \omega^n = \frac{1}{1-\omega} \sum_{n=0}^{\infty} q_n(\lambda) L_n^{(\alpha-1)}(x) \omega^n. \tag{8}$$

**Proof.** Since

$$\lim_{n\to\infty} \frac{q_{n+1}(\lambda)}{q_n(\lambda)} = \frac{1}{a}$$

and the series in (2) converges for $|\omega| < 1$, the series in the right-hand side of (8) is convergent for $|\omega| < a$.

Now, Eq. (4) gives

$$q_n(\lambda) L_n^{(\alpha-1)}(x) = q_n(\lambda) S_n(x) - q_{n-1}(\lambda) S_{n-1}(x) \tag{9}$$

and therefore

$$q_n(\lambda) S_n(x) = \sum_{i=0}^{n} q_i(\lambda) L_i^{(\alpha-1)}(x).$$

In this way, we can write

$$q_n(\lambda)S_n(x)\omega^n = \sum_{i=0}^{n} (q_i(\lambda)L_i^{(\alpha-1)}(x)\omega^i)\omega^{n-i}$$

and thus, the series

$$\sum_{n=0}^{\infty} q_n(\lambda)S_n(x)\omega^n$$

converges, because it is the Cauchy product of the two convergent series

$$\sum_{n=0}^{\infty} \omega^n = \frac{1}{1-\omega}$$

and

$$\sum_{n=0}^{\infty} q_n(\lambda)L_n^{(\alpha-1)}(x)\omega^n.$$

Moreover, we conclude

$$\sum_{n=0}^{\infty} q_n(\lambda)S_n(x)\omega^n = \frac{1}{1-\omega}\sum_{n=0}^{\infty} q_n(\lambda)L_n^{(\alpha-1)}(x)\omega^n. \qquad \square$$

We have now to distinguish $\alpha = 0$ and $\alpha \neq 0$. The generating function if $\alpha \neq 0$ will be derived in the next section. The generating function if $\alpha = 0$ is stated in the following theorem; it is the result given by Wimp and Kiesel [13] using the expression of the Laguerre–Sobolev polynomials in terms of determinants.

**Theorem 2.1.** *Let $\alpha = 0$. Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to the Sobolev inner product* (3) *with $\alpha = 0$ and normalized by the condition that the leading coefficient of $S_n$ equals the leading coefficient of $L_n^{(0)}$. Let the sequence of polynomials $\{q_n(\lambda)\}$ be defined by the recurrence relation*

$$q_{n+1}(\lambda) = (\lambda+2)q_n(\lambda) - q_{n-1}(\lambda) \tag{10}$$

*with the initial conditions $q_0(\lambda) = q_1(\lambda) = 1$.*

*Then, for $|\omega| < a < 1$,*

$$\sum_{n=0}^{\infty} q_n(\lambda)S_n(x)\omega^n = \frac{1}{(1-\omega)(1+a)} \times \left[\exp\left(-\frac{x\omega a}{1-\omega a}\right) + a\exp\left(-\frac{x\omega/a}{1-\omega/a}\right)\right], \tag{11}$$

*where*

$$a = \frac{\lambda+2-\sqrt{\lambda^2+4\lambda}}{2}.$$

**Proof.** If $\alpha = 0$ the three-term recurrence relation (7) reduces to (10). Thus, we can give an explicit representation of $q_n(\lambda)$, in fact, we have

$$q_n(\lambda) = \frac{1}{1+a}(a^n + aa^{-n}).$$

Then the theorem follows from Lemma 2.5 and (2). □

## 3. Generating function if $\alpha \neq 0$

In this section, always $\alpha > -1,\ \alpha \neq 0$. We will derive a generating function for the polynomials $\{S_n\}$ starting from relation (8). It is possible to give an explicit representation for the polynomials $q_n(\lambda)$. However, we need a generating function for the $q_n(\lambda)$ rather than the $q_n(\lambda)$ itself.

**Lemma 3.1.** *Let $\alpha > -1,\ \alpha \neq 0$ and let the polynomials $q_n(\lambda)$ be defined by the recurrence relation* (7) *with initial conditions $q_0 = q_1 = 1$. Put*

$$F(\omega) = \sum_{n=0}^{\infty} q_n(\lambda)\Gamma(n+\alpha)\frac{\omega^n}{n!}, \tag{12}$$

*with $|\omega| < a < 1$. Then*

$$F(\omega) = \Gamma(\alpha)(1 - a\omega)^{-\beta}\left(1 - \frac{\omega}{a}\right)^{-\gamma}, \tag{13}$$

*where*

$$\beta = \frac{\alpha}{1+a}, \quad \gamma = \frac{\alpha}{1+1/a} \tag{14}$$

*and*

$$a = \frac{\lambda + 2 - \sqrt{\lambda^2 + 4\lambda}}{2}.$$

**Proof.** Observe that the ratio test shows that the series in the right-hand side of (12) is convergent if $|\omega| < a < 1$.

To simplify write

$$h_n(\lambda) = \frac{q_n(\lambda)\Gamma(n+\alpha)}{n!}, \quad n \geqslant 0,$$

then

$$F(\omega) = \sum_{n=0}^{\infty} h_n(\lambda)\omega^n.$$

From the three-term recurrence relation (7) for the polynomials $q_n(\lambda)$ we obtain the recurrence relation for $h_n(\lambda)$

$$(n+1)h_{n+1}(\lambda) = \{n(\lambda+2) + \alpha\}h_n(\lambda) - (n + \alpha - 1)h_{n-1}(\lambda), \quad n \geqslant 1, \tag{15}$$

with $h_0(\lambda) = \Gamma(\alpha),\ h_1(\lambda) = \Gamma(\alpha + 1)$.

Multiply (15) with $\omega^n$ and sum over $n = 1, 2, \ldots$ then

$$F'(\omega) - h_1(\lambda) = (\lambda + 2)\omega F'(\omega) + \alpha(F(\omega) - h_0(\lambda)) - \omega^2 F'(\omega) - \alpha\omega F(\omega).$$

Hence

$$F'(\omega)\{1 - (\lambda + 2)\omega + \omega^2\} = \alpha F(\omega)(1 - \omega).$$

Observe

$$\lambda + 2 = a + \frac{1}{a},$$

then

$$F'(\omega)(\omega - a)\left(\omega - \frac{1}{a}\right) = \alpha F(\omega)(1 - \omega)$$

and

$$\frac{F'(\omega)}{F(\omega)} = -\frac{\gamma}{(\omega - a)} - \frac{\beta}{(\omega - 1/a)},$$

where

$$\beta = \frac{\alpha}{1 + a}, \quad \gamma = \frac{\alpha}{1 + 1/a}$$

and the lemma follows from $F(0) = h_0(\lambda) = \Gamma(\alpha)$. $\square$

**Remark 3.1.** Relation (15) is the recurrence relation for the Pollaczek polynomials with suitable choice of the parameters. In fact,

$$h_n = \Gamma(\alpha) P_n^{\alpha/2}\left(\frac{\lambda + 2}{2}; -\frac{\alpha}{2}, \frac{\alpha}{2}\right)$$

and Lemma 3.1 can be derived from the generating function of the Pollaczek polynomials, see [2, p. 184].

**Lemma 3.2.** *Let* $\alpha > -1$, $\alpha \neq 0$ *and* $|\omega| < a < 1$. *Then*

$$\sum_{n=0}^{\infty} q_n(\lambda) L_n^{(\alpha-1)}(x)\omega^n = \Gamma(\alpha)(1 - a\omega)^{-\beta}\left(1 - \frac{\omega}{a}\right)^{-\gamma} \sum_{l=0}^{\infty} \binom{-\beta}{l} \left(\frac{x\omega a}{1 - \omega a}\right)^l \times \sum_{m=0}^{\infty} \binom{-\gamma}{m} \left(\frac{x\omega/a}{1 - \omega/a}\right)^m \frac{1}{\Gamma(\alpha + l + m)},$$

*where* $\beta$ *and* $\gamma$ *are defined by* (14).

**Proof.** Using the explicit representation of the Laguerre polynomials (see [12, p. 101, (5.1.6)]) we get

$$\begin{aligned}
K &= \sum_{n=0}^{\infty} q_n(\lambda) L_n^{(\alpha-1)}(x)\omega^n \\
&= \sum_{n=0}^{\infty} q_n(\lambda)\omega^n \sum_{k=0}^{n} \frac{\Gamma(n+\alpha)}{(n-k)!\Gamma(\alpha+k)} \frac{(-x)^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(-x)^k \omega^k}{k!\Gamma(\alpha+k)} \sum_{n=k}^{\infty} \frac{q_n(\lambda)\Gamma(n+\alpha)}{(n-k)!}\omega^{n-k}.
\end{aligned}$$

We now apply Lemma 3.1

$$\begin{aligned}
K &= \sum_{k=0}^{\infty} \frac{(-x)^k \omega^k}{k!\Gamma(\alpha+k)} F^{(k)}(\omega) \\
&= \Gamma(\alpha) \sum_{k=0}^{\infty} \frac{(-x)^k \omega^k}{k!\Gamma(\alpha+k)} \sum_{l=0}^{k} \binom{k}{l} D^l (1-a\omega)^{-\beta} D^{k-l}\Big(1-\frac{\omega}{a}\Big)^{-\gamma} \\
&= \Gamma(\alpha)(1-a\omega)^{-\beta}\Big(1-\frac{\omega}{a}\Big)^{-\gamma} \\
&\quad \times \sum_{k=0}^{\infty} \frac{x^k \omega^k}{\Gamma(\alpha+k)} \sum_{l=0}^{k} \binom{-\beta}{l}\Big(\frac{a}{1-a\omega}\Big)^l \binom{-\gamma}{k-l}\Big(\frac{1/a}{1-\omega/a}\Big)^{k-l} \\
&= \Gamma(\alpha)(1-a\omega)^{-\beta}\Big(1-\frac{\omega}{a}\Big)^{-\gamma} \\
&\quad \times \sum_{l=0}^{\infty} \binom{-\beta}{l}\Big(\frac{x\omega a}{1-\omega a}\Big)^l \sum_{k=l}^{\infty} \binom{-\gamma}{k-l}\Big(\frac{x\omega/a}{1-\omega/a}\Big)^{k-l} \frac{1}{\Gamma(\alpha+k)}.
\end{aligned}$$

Substituting $k = l + m$ in the last series, we arrive at the lemma. □

The following lemma enables us to give the sum of the double series in Lemma 3.2.

**Lemma 3.3.** *Suppose* $\beta + \gamma \notin \{0, -1, -2, \dots\}$, *then*

$$\begin{aligned}
&\sum_{l=0}^{\infty} \binom{-\beta}{l} u^l \sum_{m=0}^{\infty} \binom{-\gamma}{m} \frac{v^m}{\Gamma(\beta+\gamma+l+m)} \\
&\quad = \frac{e^{-v}}{\Gamma(\beta+\gamma)}\, {}_1F_1(\beta;\beta+\gamma;v-u) = \frac{e^{-u}}{\Gamma(\beta+\gamma)}\, {}_1F_1(\gamma;\beta+\gamma;u-v).
\end{aligned}$$

**Proof.**

$$\sum_{l=0}^{\infty}\binom{-\beta}{l}u^l\sum_{m=0}^{\infty}\binom{-\gamma}{m}\frac{v^m}{\Gamma(\beta+\gamma+l+m)}$$
$$=\sum_{l=0}^{\infty}\binom{-\beta}{l}\frac{u^l}{\Gamma(\beta+\gamma+l)}\sum_{m=0}^{\infty}\frac{(\gamma)_m(-v)^m}{m!(\beta+\gamma+l)_m}$$
$$=\sum_{l=0}^{\infty}\binom{-\beta}{l}\frac{u^l}{\Gamma(\beta+\gamma+l)}\,{}_1F_1(\gamma;\beta+\gamma+l;-v).$$

Using Kummer's first relation

$${}_1F_1(a;c;z)=e^z{}_1F_1(c-a;c;-z), \tag{16}$$

we obtain

$$e^{-v}\sum_{l=0}^{\infty}\binom{-\beta}{l}\frac{u^l}{\Gamma(\beta+\gamma+l)}\,{}_1F_1(\beta+l;\beta+\gamma+l;v)$$
$$=\frac{e^{-v}}{\Gamma(\beta+\gamma)}\sum_{l=0}^{\infty}\frac{(\beta)_l(-u)^l}{l!(\beta+\gamma)_l}\,{}_1F_1(\beta+l;\beta+\gamma+l;v)$$
$$=\frac{e^{-v}}{\Gamma(\beta+\gamma)}\sum_{l=0}^{\infty}\frac{(-u)^l}{l!}\left(\frac{d}{dv}\right)^l{}_1F_1(\beta;\beta+\gamma;v).$$

The last series is the Taylor expansion of ${}_1F_1(\beta;\beta+\gamma;v-u)$, which proves the first assertion of the lemma. The second equality follows with (16). $\square$

**Remark 3.2.** Lemma 3.3 can also be derived from [3, Section 5.10 (1) and Section 5.7.1 (6)]. By the first relation

$$F_1\left(a,\beta,\gamma;\beta+\gamma;-\frac{u}{a},-\frac{v}{a}\right)=\left(1+\frac{v}{a}\right)^{-a}F\left(a,\beta;\beta+\gamma;\frac{-u/a+v/a}{1+v/a}\right)$$

and taking limit as $a\to\infty$ the first equality of Lemma 3.3 follows.

From Lemmas 2.5, 3.2 and 3.3 we obtain our main result. Observe that $\beta$ and $\gamma$ in (14) satisfy $\beta+\gamma=\alpha$.

**Theorem 3.1.** *Let* $\alpha>-1$, $\alpha\neq 0$. *Let* $\{S_n\}$ *denote the sequence of polynomials orthogonal with respect to the Sobolev inner product* (3), *normalized by the condition that the leading coefficient of* $S_n$ *equals the leading coefficient of* $L_n^{(\alpha)}$. *Let the sequence of polynomials* $\{q_n(\lambda)\}$ *be defined by the recurrence relation* (7) *with* $q_0(\lambda)=q_1(\lambda)=1$.

*Then, for* $|\omega| < a < 1$,

$$(1-\omega)\sum_{n=0}^{\infty} q_n(\lambda)S_n(x)\omega^n = (1-a\omega)^{-\beta}\left(1-\frac{\omega}{a}\right)^{-\gamma} e^{-v}\ {}_1F_1(\beta;\alpha;v-u)$$
$$= (1-a\omega)^{-\beta}\left(1-\frac{\omega}{a}\right)^{-\gamma} e^{-u}\ {}_1F_1(\gamma;\alpha;u-v), \tag{17}$$

*where*

$$\beta = \frac{\alpha}{1+a}, \quad \gamma = \frac{\alpha}{1+1/a}$$

*and*

$$u = \frac{x\omega a}{1-\omega a}, \quad v = \frac{x\omega/a}{1-\omega/a}, \quad a = \frac{\lambda+2-\sqrt{\lambda^2+4\lambda}}{2}. \tag{18}$$

**Remark 3.3.** If $k \geqslant 1$, then substitution of $\alpha = 0$ in $\frac{(\beta)_k}{(\alpha)_k}$ reduces it to $\frac{1}{1+a}$. Hence, substitution of $\alpha = 0$ in ${}_1F_1(\beta;\alpha;v-u)$ gives

$$1 + \frac{1}{1+a}\sum_{k=1}^{\infty}\frac{1}{k!}(v-u)^k = \frac{1}{1+a}e^{v-u} + \frac{a}{1+a}$$

and we arrive at (11), the result of Wimp and Kiesel.

**Remark 3.4.** For $\lambda = 0$, we have $q_n = 1$ for all $n \geqslant 0$, $S_n = L_n^{(\alpha)}$, $a = 1$, so the confluent hypergeometric function reduces to ${}_1F_1(\frac{\alpha}{2};\alpha;0) = 1$ and the theorem reduces to (2), the generating function of the classical Laguerre polynomials.

## 4. Generalizations

The results of the preceding sections can be generalized to Sobolev inner products of the form

$$(f,g)_S = \int_0^{+\infty} f(x)g(x)\,d\psi_0(x) + \lambda\int_0^{+\infty} f'(x)g'(x)x^{\alpha}e^{-x}\,dx, \tag{19}$$

with $\lambda > 0$, $\alpha \geqslant 0$ and

(a) if $\alpha = 0$, then $d\psi_0(x) = e^{-x}\,dx + M\delta(0)$, with $M \geqslant 0$;
(b) if $\alpha \neq 0$, then $d\psi_0(x) = (x-\xi)x^{\alpha-1}e^{-x}\,dx$, with $\xi \leqslant 0$.

The pair $\{d\psi_0(x), x^{\alpha}e^{-x}\,dx\}$ is a coherent pair of Laguerre type I studied in [10].

Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to (19) with leading coefficient of $S_n$ equal to the leading coefficient of $L_n^{(\alpha)}$. Relations (4) and (6) are still satisfied; for a proof we refer to [10].

**Lemma 4.1.** *There exist positive constants $a_n$ depending on $\alpha$, $\lambda$ and $M$ or $\xi$, such that*

$$L_n^{(\alpha-1)}(x) = S_n(x) - a_{n-1}S_{n-1}(x), \quad n \geqslant 1; \tag{20}$$

*the sequence $\{a_n\}$ is convergent and*

$$a = \lim_{n\to\infty} a_n = \frac{\lambda+2-\sqrt{\lambda^2+4\lambda}}{2} < 1.$$

Define as in Section 2 the $\{q_n(\lambda)\}$ by

$$q_0(\lambda) = 1, \quad q_{n+1}(\lambda) = \frac{q_n(\lambda)}{a_n}, \quad n \geqslant 0.$$

Then (20) implies

$$q_n(\lambda)L_n^{(\alpha-1)}(x) = q_n(\lambda)S_n(x) - q_{n-1}(\lambda)S_{n-1}(x), \quad n \geqslant 1.$$

This is the starting formula (9) in the proof of Lemma 2.5. Hence, Lemma 2.5 is still satisfied.

The recurrence relation for the $a_n$ in (20), however, is somewhat different from the recurrence relation in Lemma 2.2. We distinguish $\alpha = 0$ and $\alpha > 0$.

**Theorem 4.1.** *Let $\alpha = 0$. Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to* (19) *with $d\psi_0(x) = e^{-x}\,dx + M\delta(0)$ where $M \geqslant 0$ and the leading coefficient of $S_n$ be equal to the leading coefficient of $L_n^{(0)}$. Let the sequence of polynomials $\{q_n(\lambda)\}$ be defined by*

$$q_{n+1}(\lambda) = (\lambda+2)q_n(\lambda) - q_{n-1}(\lambda), \quad n \geqslant 1, \tag{21}$$

*with the initial conditions $q_0(\lambda) = 1,\ q_1(\lambda) = M+1$.*

*Then, for $|\omega| < a < 1$,*

$$\sum_{n=0}^{\infty} q_n(\lambda)S_n(x)\omega^n = \frac{1}{1-\omega}\left[A\exp\left(-\frac{x\omega a}{1-\omega a}\right) + B\exp\left(-\frac{x\omega/a}{1-\omega/a}\right)\right],$$

*where*

$$A = \frac{1}{1+a} - M\frac{a}{1-a^2}, \quad B = \frac{a}{1+a} + M\frac{a}{1-a^2}. \tag{22}$$

**Proof.** The recurrence relation for the $a_n$ in (20) reads (see [10])

$$a_n = \frac{1}{2+\lambda-a_{n-1}}, \quad n \geqslant 1, \quad a_0 = \frac{1}{M+1},$$

then the recurrence relation for the $q_n$ becomes (21) with $q_0(\lambda) = 1,\ q_1(\lambda) = M+1$. The recurrence relation (21) can be solved explicitly and

$$q_n(\lambda) = Aa^n + Ba^{-n}$$

with $A$ and $B$ given by (22). Then (2) and Lemma 2.5 give the desired result. □

**Remark 4.1.** For $M=0$ Theorem 4.1 reduces to Theorem 2.1, the result of Wimp and Kiesel [13].

We now turn to the case $\alpha>0$, $d\psi_0(x)=(x-\xi)x^{\alpha-1}e^{-x}\,dx$, with $\xi\leqslant 0$. The recurrence relation for the $a_n$ in (20) reads (see [10])

$$a_n=\frac{n+\alpha}{n(2+\lambda)+\alpha-\xi-na_{n-1}},\quad n\geqslant 1$$

and $a_0=\frac{\alpha}{\alpha-\xi}$.

This implies the recurrence relation for the $q_n(\lambda)$:

$$(n+\alpha)q_{n+1}(\lambda)=\{n(\lambda+2)+\alpha-\xi\}q_n(\lambda)-nq_{n-1}(\lambda),\quad n\geqslant 1, \tag{23}$$

with initial conditions $q_0(\lambda)=1,\ q_1(\lambda)=1-\frac{\xi}{\alpha}$. Lemma 3.1 on the generating function of the $\{q_n(\lambda)\}$ has to be modified.

**Lemma 4.2.** *Let $\alpha>0$ and let the polynomials $q_n(\lambda)$ be defined by the recurrence relation* (23) *with initial conditions $q_0(\lambda)=1,\ q_1(\lambda)=1-\frac{\xi}{\alpha}$. Put*

$$F(\omega)=\sum_{n=0}^{\infty}q_n(\lambda)\Gamma(n+\alpha)\frac{\omega^n}{n!}$$

*with $|\omega|<a<1$. Then*

$$F(\omega)=\Gamma(\alpha)(1-a\omega)^{-\beta}\left(1-\frac{\omega}{a}\right)^{-\gamma}, \tag{24}$$

*where*

$$\beta=\frac{\alpha}{1+a}+\frac{\xi a}{1-a^2},\quad \gamma=\frac{\alpha}{1+1/a}-\frac{\xi a}{1-a^2} \tag{25}$$

*and*

$$a=\frac{\lambda+2-\sqrt{\lambda^2+4\lambda}}{2}.$$

**Proof.** With

$$h_n(\lambda)=\frac{q_n(\lambda)\Gamma(n+\alpha)}{n!},\quad n\geqslant 0,$$

relation (23) is transformed in

$$(n+1)h_{n+1}(\lambda)=\{n(\lambda+2)+\alpha-\xi\}h_n(\lambda)-(n+\alpha-1)h_{n-1}(\lambda),\quad n\geqslant 1,$$

with $h_0(\lambda)=\Gamma(\alpha),\ h_1(\lambda)=\Gamma(\alpha)(\alpha-\xi)$.

This implies

$$\begin{aligned}F'(\omega)-h_1(\lambda)={}&(\lambda+2)\omega F'(\omega)+(\alpha-\xi)(F(\omega)-h_0(\lambda))\\&-\omega^2F'(\omega)-\alpha\omega F(\omega).\end{aligned}$$

Hence

$$F'(\omega)\{1-(\lambda+2)\omega+\omega^2\}=F(\omega)(\alpha-\xi-\alpha\omega).$$

Then

$$\frac{F'(\omega)}{F(\omega)} = -\frac{\gamma}{(\omega - a)} - \frac{\beta}{(\omega - 1/a)},$$

where $\beta$ and $\gamma$ are defined in (25). With $F(0) = \Gamma(\alpha)$ we arrive at (24). $\square$

Relation (24) equals (13) with the $\beta$ and $\gamma$ in (14) replaced by their values in (25). Observe that they still satisfy $\beta + \gamma = \alpha$. The calculations in the proof of Lemma 3.2 do not depend on the special values $\beta$ and $\gamma$. So Lemma 3.2 is still satisfied with the values of $\beta$ and $\gamma$ given in (25). Finally, we arrive at the generating function for $S_n$ stated in the following theorem.

**Theorem 4.2.** *Let $\alpha > 0$. Let $\{S_n\}$ denote the sequence of polynomials orthogonal with respect to* (19) *with $d\psi_0(x) = (x - \xi)x^{\alpha-1}e^{-x}\,dx$, where $\xi \leqslant 0$, and the leading coefficient of $S_n$ be equal to the leading coefficient of $L_n^{(\alpha)}$. Let the sequence of polynomials $\{q_n(\lambda)\}$ be defined by* (23) *with $q_0(\lambda) = 1$, $q_1(\lambda) = 1 - \frac{\xi}{\alpha}$. Then, for $|\omega| < a < 1$, the generating function relation* (17) *is satisfied with $\beta$ and $\gamma$ given by* (25) *and $u, v$ and $a$ given by* (18).

## References


[1] J. Brenner, Über eine Erweiterung des Orthogonalitätsbegriffes bei Polynomen, in: G. Alexits, S.B. Stechkin (Eds.), Constructive Theory of Functions, Akadémiai Kiadó, Budapest, 1972, pp. 77–83.

[2] T.S. Chiara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger, G. Tricomi, Higher Transcendental Functions, The Bateman Manuscript Project, Vol. 1, McGraw-Hill, New York, 1953.

[4] A. Iserles, P.E. Koch, S.P. Nørsett, J.M. Sanz-Serna, On polynomials orthogonal with respect to certain Sobolev inner products, J. Approx. Theory 65 (1991) 151–175.

[5] F. Marcellán, M. Alfaro, M.L. Rezola, Orthogonal polynomials on Sobolev spaces: old and new directions, J. Comput. Appl. Math. 48 (1993) 113–131.

[6] F. Marcellán, H.G. Meijer, T.E. Pérez, M.A. Piñar, An asymptotic result for Laguerre–Sobolev orthogonal polynomials, J. Comput. Appl. Math. 87 (1997) 87–94.

[7] F. Marcellán, T.E. Pérez, M.A. Piñar, Laguerre–Sobolev orthogonal polynomials, J. Comput. Appl. Math. 71 (1996) 245–265.

[8] A. Martínez-Finkelshtein, J.J. Moreno-Balcázar, T.E. Pérez, M.A. Piñar, Asymptotics of Sobolev orthogonal polynomials for coherent pairs of measures, J. Approx. Theory 92 (1998) 280–293.

[9] H.G. Meijer, A short history orthogonal polynomials in a Sobolev space, I, The non-discrete case, Nieuw. Arch. Wisk. (4) 14 (1996) 93–112.

[10] H.G. Meijer, T.E. Pérez, M.A. Piñar, Asymptotics of Sobolev orthogonal polynomials for coherent pairs of Laguerre type, J. Math. Anal. Appl. 245 (2000) 528–546.

[11] F.W. Schäfke, G. Wolf, Einfache verallgemeinerte klassische orthogonalpolynome, J. Reine Angew. Math. 262/263 (1973) 339–355.

[12] G. Szegő, Orthogonal Polynomials, 4th Edition, American Mathematical Society, Colloquium Publications, Vol. 23, American Mathematical Society, Providence, RI, 1975.

[13] J. Wimp, H. Kiesel, Some properties of the Sobolev–Laguerre polynomials, J. Comput. Appl. Math. 99 (1998) 401–412.